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Structure in supersymmetric Yang-Mills theory

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Abstract: We show that requiring sixteen supersymmetries in quantum mechanical gauge theory implies the existence of a web of constrained interactions. Contrary to conventional wisdom, these constraints extend to arbitrary orders in the momentum expansion.

Keywords: Nonperturbative Effects, Extended Supersymmetry.

Contents

1. Introduction

Supersymmetry is a truly remarkable symmetry. What is perhaps more surprising is that we still do not understand the full extent of the constraints imposed by supersymmetry on field theory and string theory. The goal of this paper is to further unravel these constraints. We will consider maximally supersymmetric Yang-Mills theory with 16 real supersymmetries.

The simplest case to analyze is quantum-mechanical supersymmetric Yang-Mills theory. This is the theory that describes the low-energy dynamics of D0-branes in type IIA string theory. In the past, the techniques used to prove non-renormalization results in this theory have generalized to both higher-dimensional field theory and string theory. We can hope that the same will be true of this analysis.

Our interest is in the effective action describing the physics on the Coulomb branch of this theory. Again for simplicity, let us restrict to the rank 1 case; for example, SU(2) broken to U(1). The effective action can then be thought of as describing the dynamics of a supersymmetric particle in 9 dimensions. The position of the particle is determined by 9 scalar fields, $x^i(t)$, whose superpartners are 16 fermions, $\psi_a(t)$.

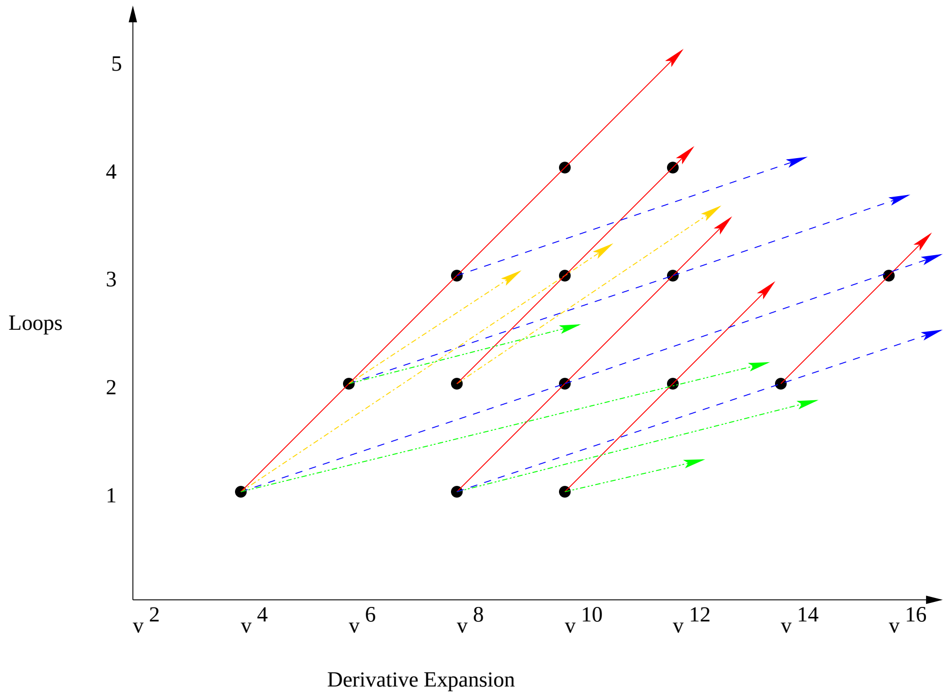

Figure 1: A depiction of the web of induced couplings.

The effective action involves couplings constructed from (x^i, ψ_a) and derivatives of these fields. To any coupling, we can assign an order, denoted n, given by

$$n = n_\partial + \frac{1}{2} n_f \, , \tag{1.1}$$

where n_∂ is the number of derivatives while n_f is the number of fermions. The order measures the relevance of the coupling at low-energies. Terms with more derivatives are less relevant at low-energies.

Because of the freedom to perform field redefinitions, the form of the effective action is ambiguous. However, as we will show in section 2.2, there is a particularly nice choice of fields. In terms of these fields, at the lowest order with $n = 2$, the action is unique taking the free-particle form [1]

$$S_1 = \int dt \, \frac{1}{2} \left(v^2 + i\psi\dot{\psi} \right) , \tag{1.2}$$

where $v = \partial_t x$. All of the remaining couplings in the effective action are constructed in terms of (v, ψ) only with no higher time derivatives. This generalizes an observation of [2] employed in [3] in a study of the $O(v^4)$ terms. This simplification is special to quantum mechanics, although it should have some analogue in higher-dimensional field theories.

At low orders in the derivative expansion, there are non-renormalization theorems. Terms of $O(v^4)$ are only generated at 1-loop [1, 4, 3], while terms of $O(v^6)$ are only generated at 2-loops [5]. There are no known results beyond $O(v^6)$. The usual intuition associated to this breakdown at $O(v^8)$ is the (heuristic) argument that these terms involve integrals over all of superspace, while $O(v^4)$ and $O(v^6)$ terms only involve integrals over a fraction of superspace. The latter interactions are therefore special. This argument is heuristic because there is no superspace construction that keeps manifest all 16 real supersymmetries. What we will find is that this intuition is actually incorrect. To some extent, we already knew this from an analysis of higher rank theories [6, 7], but we will find that this is true even for rank 1.

Our results are most easily explained via figure 1.[1] What we will primarily study is a particular spin-spin coupling at $O(v^{2n})$,

$$g_4^{(n,1)}(r) v^{2n-2} x_i x_j (\psi\gamma^{ik}\psi)(\psi\gamma^{jk}\psi)\,. \tag{1.3}$$

In principle, this coupling can be generated at any order in perturbation theory. Supersymmetry, however, imposes a much more rigid structure: this coupling is never generated at 1-loop, but can be generated at higher loops. In terms of figure 1, this spin-spin coupling is only generated at points through which some ray passes. It is never generated at any anchor point. An anchor point is a point which only originates rays. For example, consider the anchor located at coordinates $(v^4, 1)$. The existence of this point tells us that $O(v^4)$ terms are generated at 1-loop. Since this is an anchor point, the spin-spin coupling given in (1.3) is not generated. This is in accord with both an explicit computation [8] at 1-loop, and prior supersymmetry arguments [4, 3, 9]. However, the existence of 1-loop $O(v^4)$ terms generates a line of slope 1 in the figure. The next point on the line is at $(v^6, 2)$. A 2-loop spin-spin coupling is induced at this order with a coefficient that is predicted in terms of the $O(v^4)$ interactions. More precisely, in terms of the coefficient of the v^4/r^7 interaction. Effectively, the $O(v^6)$ spin-spin coupling is sourced by the $O(v^4)$ terms; had they been absent, there would be no $O(v^6)$ spin-spin coupling. This is not particularly remarkable because we already know that all terms at $O(v^6)$ are sourced by the $O(v^4)$ interactions [5].

This gets much more interesting when we go to the next point on the line at $(v^8, 3)$. A 3-loop exact spin-spin coupling is induced at this order which corresponds to,

$$g_2^{(4,1)} \sim \frac{1}{r^{25}}\,. \tag{1.4}$$

This is regardless of whether any other coupling, for example the v^8 interaction, is generated at other orders in perturbation theory. Indeed, we can predict the numerical coefficient of this spin-spin coupling from our knowledge of the v^4 and v^6 interactions. Continuing along the line, we see that an $O(v^{10})$ spin-spin coupling is induced at 4-loops. The coefficient for this interaction can be predicted in terms of the coefficient of the 3-loop contribution to the v^8 interaction, and so on.

This brings us to a critical open issue. Let us return to the terms of $O(v^8)$. We will argue that a particular coupling is determined exactly at this order; namely, the spin-spin coupling (1.3). It is also likely that there are additional couplings that can be determined using similar arguments. Using the notation described in appendix A, the 12 fermion coupling

$$v_i v_j * \left\{ (\psi\gamma^{ik}\psi)(\psi\gamma^{jk}\psi) \right\}, \tag{1.5}$$

and the couplings $(A.5)$ and $(A.6)$ are natural candidates.

[1] This figure is best seen with a color viewer!

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Are these couplings, in conjunction with symmetry, sufficient to completely determine all 3-loop terms at $O(v^8)$? If so, this would mean that there is no arbitrary 3-loop solution to the supersymmetry constraints at $O(v^8)$. We can hope that the answer is yes, but demonstrating this will require understanding the full conditions imposed by invariance under supersymmetry; perhaps, including closure of the supersymmetry algebra. Because our nice choice of fields enormously simplifies the form of the effective action, it appears to me that this question can now be fully answered for this theory.

Returning to figure 1, we see that there are other lines of slope 1. Each of these lines is induced by the existence of the 1-loop $O(v^4)$ terms. If there is any other non-zero v^{2n} interaction then a ray of slope 1 will emanate from that point on the diagram. For example, let us suppose there are non-vanishing 1 and 2-loop v^8 terms as depicted in figure 1. These terms are generically expected to be non-vanishing, and in fact, the 2-loop term is known to be non-vanishing [10]. They then correspond to new anchor points in the diagram from which rays of slope 1 extend. Since they are anchor points, there is no 1-loop or 2-loop spin-spin coupling at $O(v^8)$, but there are induced 2-loop and 3-loop contributions to the spin-spin coupling at $O(v^{10})$ whose coefficients can be predicted.

Further, the new anchor points give rise to new lines with slopes less than 1. This comes about in the following way. The $O(v^{2n})$ terms generate corrections to the supersymmetry transformations (of the same loop order) that allow you to move $(n-1)$ steps in the derivative expansion. So the $O(v^4)$ corrections connect terms in the derivative expansion that differ by $O(v^2)$, or 1 step. The $O(v^8)$ terms, however, connect terms in the derivative expansion that differ by $O(v^6)$, or 3 steps. Therefore, the 1-loop $O(v^8)$ term generates a 2-loop $O(v^{14})$ term. Hence, there are lines of slope 1/3 in the diagram. On the other hand, the 2-loop $O(v^8)$ term generates a 4-loop $O(v^{14})$ term which explains the lines of slope 2/3. Lastly, the 1-loop $O(v^{10})$ anchor point (should those terms be non-vanishing as we expect) gives rise to lines of slope 1/4 for exactly the same reasons. These rays with different slopes extend from every node in figure 1 although, for clarity, only some of the rays are actually depicted.

From the perspective of a perturbative field theorist, this web of interactions must look bewilderingly complicated. With the aid of symmetry, however, we will see that the structure has remarkably simple origins. In fact, there is more structure than a single determined spin-spin coupling. The most studied terms in the effective action have the form,

$$g_0^{(n)}(r)v^{2n} + g_2^{(n)}(r)v^{2n-2}x^i v^j \psi\gamma^{ij}\psi + \cdots , \tag{1.6}$$

where $g_2^{(n)}(r)$ is the coefficient of the spin coupling. This spin coupling at $O(v^4)$ has been studied in [8] and [11]–[14]. For general n, we do not yet know how to separately fix $g_0^{(n)}$ and $g_2^{(n)}$. However, supersymmetry does fix the combination

$$g_2^{(n)} + \frac{i\{g_0^{(n)}\}'}{2r}, \tag{1.7}$$

in terms of more relevant interactions, just like the spin-spin coupling. This is even true for anchor points for which,

$$g_2^{(n)} + \frac{i\{g_0^{(n)}\}'}{2r} = 0\,, \tag{1.8}$$

and the spin coupling is fixed in terms of the v^{2n} interaction. There are plenty of relations of this sort between interactions in the effective action.

There are many directions to explore. As I stressed earlier, the complete set of supersymmetry constraints can, in principle, now be determined for this theory. The extension to higher rank, higher dimensions, and less supersymmetry will involve novel issues, and hopefully provide novel results. For example, it is worth stressing that these results are actually non-perturbative. When applied to Yang-Mills in three dimensions or type IIB string theory, we should be able to learn about instanton corrections to special interactions [15]–[19]. The Matrix theory [20] interpretation of these relations should teach us something about M theory. Lastly, extending this analysis to N=4 Yang-Mills in four dimensions should also prove interesting in light of the recent conjecture about the structure of certain perturbative gluon ampitudes [21].

2. D0-Brane Dynamics

2.1 Some preliminaries

The low-energy degrees of freedom describing the dynamics of D0-branes consist of 9 bosons, x^i, transforming in the vector representation of the R-symmetry group $Spin(9)$. In addition, there are 16 real fermions, ψ_a, transforming in the spinor representation. The bosons have mass dimension $[x] = 1$ while the fermions have mass dimension $[\psi] = 3/2$. The Yang-Mills coupling constant, g^2, has mass dimension 3.

The effective action for these degrees of freedom takes the schematic form,

$$S = \int dt\, \left(f_1(r)v^2 + f_2(r)v^4 + f_3(r)v^6 + \ldots \right), \tag{2.1}$$

where $f_k v^{2k}$ simply represents all possible terms in the momentum expansion of appropriate order. We define the $Spin(9)$ gamma matrices by the relation:

$$\left\{\gamma^i, \gamma^j\right\} = 2\delta^{ij}\,. \tag{2.2}$$

The supersymmetry transformations can then be expressed in the form,

$$\begin{aligned} \delta x^i &= -i\epsilon\gamma^i\psi + \epsilon N^i\psi \\ \delta\psi_a &= \left(\gamma^i v^i \epsilon\right)_a + (M\epsilon)_a\,. \end{aligned} \tag{2.3}$$

We have lumped all the complicated corrections to the free-particle supersymmetry transformations into N and M.

What is known about the effective action can be summarized as follows: at order 2, the action is unique taking the free-particle form (1.2). At order 4, the action takes the form

$$S_2 = \int dt\, \left(f_2^{(0)}(r)v^4 + \cdots + f_2^{(8)}(r)\psi^8 \right). \tag{2.4}$$

Again, the notation is schematic, and includes interactions involving accelerations a, $\dot{a}$, $\dot{\psi}$, $\ddot{\psi}$ terms etc. The technique developed in [1] tells us that the $f_2^{(8)}(r)\psi^8$ interactions are 1-loop exact. In a series of papers [4, 14] and [22]–[24], culminating in [3], this result was shown to imply the non-renormalization of all other terms at this order, as conjectured in [1].

Lastly, at order 6, the argument of [5] shows that the $f_3^{(12)}(r)\psi^{12}$ interactions are 2-loop exact. These interactions are really determined by the terms in S_2. One way to view this result is that the ψ^{12} interactions are slaves of the ψ^8 terms needed to obtain a closed supersymmetry algebra. Although it has yet to be explicitly checked, we believe that all terms at order 6 are related by supersymmetry to the $f_3^{(12)}(r)\psi^{12}$ interactions, and are therefore 2-loop exact. In fact, this will become clear in light of our subsequent analysis.

The argument given in [1] breaks down at order 8 for a simple reason. Consider the 'top form' interaction,

$$f_4(r)\psi^{16}\,, \tag{2.5}$$

and vary f_4. At order 4 and order 6, the analogous variation of $f_2^{(8)}(r)\psi^8$ and $f_3^{(12)}(r)\psi^{12}$ leads to a non-vanishing 9 fermion and 13 fermion term, respectively. Since these variations involve no space-time derivatives, they must either be cancelled or vanish. This argument leads to the non-renormalization results. However, in this case, the variation automatically vanishes since we only have 16 fermions. The order 8 interactions are therefore not expected to be special in anyway, and could in principle be generated at any order in perturbation theory (and in higher dimensions, non-perturbatively).

2.2 Simplifying the action

Can we say more? To proceed, we first extend an observation of [2, 3]. Consider all terms of order $2n$ where $n > 1$. Integrating by parts allows us to express the action in a special form:

$$S_n = \int \left(f_n(x,v,\psi) + a^i k_i + \dot{\psi}_a h^a\right). \tag{2.6}$$

Here f_n contains no accelerations or higher time derivatives of x, and no time derivatives acting on ψ. All such terms are lumped into k_i and h^a.

Noting the special form of S_1 given in (1.2), we see that the field redefinition

$$x^i \to x^i + k^i\,, \tag{2.7}$$

$$\psi_a \to \psi_a + \frac{i}{2}h_a\,, \tag{2.8}$$

removes the acceleration and $\dot{\psi}$ terms while leaving terms of order less than $2n$ invariant. By induction, we can remove all acceleration and $\dot{\psi}$ terms leaving an action, aside from S_1, which depends only on x, v, ψ.

This is a nice simplification which teaches us that with this choice of fields, the action has the schematic form

$$S = \int dt\,\left(g_0(v,x) + g_2(v,x)x^i v^j \psi\gamma^{ij}\psi + \cdots + g_{16}(v,x)\psi^{16}\right), \tag{2.9}$$

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where the scalar functions, g_m, depend on the $Spin(9)$ invariants (v^2, x^2, $x \cdot v$). Note that the action (2.9) is not a momentum expansion! Each g_{2m} appears with some combination of $2m$ fermions, but contains terms of all orders in velocity. There are unique g_2 and g_{16} functions because there are unique non-vanishing 2 fermion and 16 fermion structures. For other choices of m, there can be many independent $2m$ fermion structures each appearing with its own $g_{2m}^{(i)}$ function. We will try to avoid delving into those details until later.

As a final simplification, we can choose each g_{2m} to depend only on (v^2, x^2) and not on $(x \cdot v)$. The argument goes as follows: consider any term of the form

$$g_{2m}(v^2, r)(x \cdot v)^k \left(x^{i_1} \ldots x^{i_l} v^{j_1} \ldots v^{j_p}\right) T^{a_1 \ldots a_{2m}}_{i_1 \ldots i_l, j_1 \ldots j_p} \psi_{a_1} \ldots \psi_{a_{2m}}$$

where T is some structure constructed from γ matrices. Up to the introduction of acceleration and v^2 terms, we can make the substitution

$$g_{2m}(v^2, r)(x \cdot v)^k \rightarrow \frac{1}{2}\partial_t \{ \widetilde{g}_{2m} x^2 (x \cdot v)^{k-1} \},$$

where we choose $\widetilde{g}_{2m}$ to satisfy

$$\left(1 + \frac{1}{2} r \partial_r\right) \widetilde{g}_{2m} = g_{2m} \,. \tag{2.10}$$

This equation is always solvable. After this substitution, we can integrate by parts to leave only terms depending on $(x \cdot v)^{k-1}$, a, or $\dot{\psi}$. The a and $\dot{\psi}$ terms can be field redefined away, and the procedure repeated until only v^2 type terms remain. This is not an essential simplification, but it does make the algebra cleaner.

2.3 Demonstrating non-genericity

Let us begin by supposing that the theory consists only of the free particle terms in S_1 and the terms of order $2n$ in S_n so

$$S = S_1 + S_n \,. \tag{2.11}$$

We will treat all the intervening terms with orders less than $2n$ as sources, but first we need to understand the homogeneous solution to the supersymmetry constraints.

With the simplifications described in section 2.2, we can express the terms in S_n in the form,

$$g_0^{(n)}(r) v^{2n} + g_2^{(n)}(r) v^{2n-2} x^i v^j \psi \gamma^{ij} \psi + \cdots \,. \tag{2.12}$$

These terms will induce corrections to the supersymmetry transformations. Let us expand the supersymmetry generators δ_a in a series

$$\delta_a = \delta_a^1 + \delta_a^2 + \cdots \,,$$

where the δ_a^n term is induced by terms of order $2n$. The invariance condition then reduces to the statement that

$$\delta^n S_1 + \delta^1 S_n = 0 \,.$$

The terms from $\delta^n S_1$ are particularly nice taking the form,

$$-a^i \epsilon N^i \psi - i\dot{\psi} M \epsilon \,, \tag{2.13}$$

where N and M are defined in (2.3). It is key that these terms involve either a or $\dot{\psi}$. The strategy then is to vary (2.12) and separate out the $a, \dot{\psi}$ terms.

For the 1 fermion terms in the variation of (2.12), it is an easy exercise to separate out the $(a, \dot{\psi})$ terms from the rest. The resulting equations imply that

$$\begin{aligned} g_2^{(n)} &= -\frac{i\{g_0^{(n)}\}'}{2r} \,, & (2.14)\\ \epsilon N_0^i \psi &= (2n-2) i g_0^{(n)} v^{2n-4} v^i v^k (\epsilon \gamma^k \psi) + i g_0^{(n)} v^{2n-2} (\epsilon \gamma^i \psi) \,, & (2.15)\\ M_0 \epsilon &= (2n-1) g_0^{(n)} v^{2n-2} v^k \gamma^k \epsilon \,. & (2.16) \end{aligned}$$

The subscript on N and M denotes the number of fermions in the terms under consideration. When we need to distinguish terms in (N, M) generated at $O(v^{2n})$, we will again use the superscript notation $(N^{(n)}, M^{(n)})$. The noteworthy features of (2.15) and (2.16) are that both N_0 and M_0 depend on only 1 gamma matrix, and that M_0 has the same form as the free-particle result (2.3).

To see something interesting, we need to consider the 4 fermion terms in S_n. As described in appendix A, there are three such structures which take the form

$$\begin{aligned} S_n = \cdots &+ g_4^{(n,1)}(r) v^{2n-2} x_i x_j (\psi \gamma^{ik} \psi)(\psi \gamma^{jk} \psi) + g_4^{(n,2)}(r) v^{2n-4} v_i v_j (\psi \gamma^{ik} \psi)(\psi \gamma^{jk} \psi) + \\ &+ g_4^{(n,3)}(r) v^{2n-4} x_i x_j v_k v_l (\psi \gamma^{ik} \psi)(\psi \gamma^{jl} \psi) + \cdots \,. \end{aligned} \tag{2.17}$$

This looks messy, but the observation we need to make only concerns the $g_4^{(n,1)}$ term. We want to study the supersymmetry variation of (2.17) into 3 fermions, but this has a piece that looks like

$$4 g_4^{(n,1)}(r) v^{2n-2} x^i x^j v_p (\epsilon \gamma^{pik} \psi)(\psi \gamma^{jk} \psi) \,.$$

No other term obtained by varying the structures in (2.17) contains a 5 gamma piece. It is also easy to see that varying the g_2 structure into 3 fermions never gives a 5 gamma term. Lastly, the terms from varying S_1 contain either a or $\dot{\psi}$ and do not mix with this term. We can therefore conclude that

$$g_4^{(n,1)} = 0 \,. \tag{2.18}$$

This obervation was already made for the case $n = 2$ in [4]. Here we see that it is true to all orders in the derivative expansion. The vanishing of this coupling implies that the action is non-generic!

2.4 Sources?

While this coupling is absent for the homogeneous solution, it might be generated by sources. To determine whether this is the case, we need to compute the N^i and M terms inductively. Fortunately, some of the required computations have already been performed [3]. The generalization of those results appears in appendix B for general n. First note that in the absence of sources, the coefficient functions for the homogeneous solution satisfy the

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relations,

$$ig_2^{(n)} + 4g_4^{(n,2)} - 2\widetilde{g}_4^{(n,1)} = 0\,, \tag{2.19}$$

$$4g_4^{(n,1)} + 4g_4^{(n,3)} + \frac{i}{r}\{g_2^{(n)}\}' = 0\,, \tag{2.20}$$

$$g_4^{(n,1)} = 0\,. \tag{2.21}$$

Since the first few orders in the momentum expansion are special, we will now proceed order by order.

2.4.1 Terms of $O(v^4)$

The first case is the $O(v^4)$ terms for which $n = 2$. In this case, $g_4^{(2,1)}$ must vanish since there are no sources,

$$g_4^{(2,1)}(r) = 0\,, \tag{2.22}$$

but we need to determine N_2^i and M_2. These terms are listed in Appendix B. What we want to know is whether any term in N or M can source the coupling $g_4^{(3,1)}$ at $O(v^6)$. A quick perusal of the expression $(B.4)$ restricting to $n = 2$ and setting $g_4^{(2,1)} = 0$ shows us that only one term in M_2 is relevant

$$M_2\epsilon = \cdots + g_2^{(2)} v^2 (x^i \psi\gamma^{ij}\psi)(\gamma^j\epsilon)\,. \tag{2.23}$$

This term sources equation (2.21) via the variation of the spin coupling,

$$2g_2^{(2)} v^2 x^i v^j \psi\gamma^{ij} M_2^{(2)}\epsilon\,,$$

in $\delta^2 S_2$, which has a 5-gamma piece that mixes with the $g_4^{(3,1)}$ coupling at $O(v^6)$. This is the only source with the right gamma matrix structure in $\delta^2 S_2$. The homogeneous solution is therefore modified by this source,

$$4g_4^{(3,1)} - 2\{g_2^{(2)}\}^2 = 0\,. \tag{2.24}$$

The good news is that the previously vanishing spin-spin coupling is induced at $O(v^6)$ by the $O(v^4)$ terms. Since the $O(v^4)$ terms are 1-loop exact, this coupling is 2-loop exact, which is no great surprise since we know that all couplings at $O(v^6)$ are 2-loop exact. However, it is important to note that this coupling is 2-loop exact regardless of how the other $O(v^6)$ couplings are renormalized. No independent argument is required. It should be possible to check the relation (2.24) directly using the techniques of [8] combined with the computations of [25, 26].

2.4.2 Terms of $O(v^6)$

Let us now ask whether the spin-spin coupling at $O(v^8)$ is similarly sourced. The potential sources are generated in two ways: either from contributions to the supersymmetry transformations generated at $O(v^6)$ which act on terms of $O(v^4)$, or from contributions generated at $O(v^4)$ which act on terms of $O(v^6)$. Said differently, the sources come from terms in the variations,

$$\delta^2 S_3 + \delta^3 S_2\,.$$

To determine the contributions from $\delta^3 S_2$, we need to learn about the terms in $N^{i(3)}$ and $M^{(3)}$. which are generated at $O(v^6)$. First note that the homogeneous solution for the coefficient functions (2.12) at $O(v^6)$ is now modified by the $O(v^4)$ sources, as we already saw in (2.24). Relation (2.14) becomes

$$g_2^{(3)} = -\frac{i\{g_0^{(3)}\}'}{2r} + \frac{2i}{r}\frac{d}{dr}\{g_0^{(2)}\}^2 \,. \tag{2.25}$$

For the following argument, it turns out that we do not need the explicit form of $N_0^{i(3)}$ and $M_0^{(3)}$ generated at $O(v^6)$. To see why we first need to realize that these terms have the same form as the homogeneous solution (2.15) and (2.16), but with different coefficient functions. This is not hard to show explicitly, and is also true just on general grounds. So we need to ask whether an $(N_0^{i(3)}, M_0^{(3)})$ of this form can possibly mix with the $g_4^{(4,1)}$ coupling by appearing in the variation $\delta^3 S_2$.

This would have been the case had $g_4^{(2,1)}$ not vanished via the $M_0^{(3)}$ to the supersymmetry transformations. Since this coupling does vanish, there is no mixing and we need not worry about the explicit form of these terms. The same cannot be said for $(N_2^{i(3)}, M_2^{(3)})$ which can, in principle, mix with the $g_4^{(4,1)}$ coupling.

Of the terms in the homogeneous solution appearing in (B.4), only two are relevant.

$$M_2\epsilon = \cdots - 2i\widetilde{g}_4^{(3,1)} x^2 v^4 x^i (\psi\gamma^{ik}\psi)(\gamma^k\epsilon) + g_2^{(3)} v^4 (x^i\psi\gamma^{ij}\psi)(\gamma^j\epsilon)\,. \tag{2.26}$$

Both these coefficient functions are modified by sources according to (2.24) and (2.25). These terms acting on the 2-fermion coupling at $O(v^4)$ can source the spin-spin coupling at $O(v^8)$.

Are there any relevant inhomogeneous pieces of $(N_2^{i(3)}, M_2^{(3)})$? These inhomogeneous terms are sourced from the variation $\delta^2 S_2$. Let us examine these corrections term by term. The first terms come, schematically, from the variation

$$\delta(g_0^{(2)} v^4)$$

using $N_2^{i(2)}$. This variation has terms of the form $x^i\epsilon N_2^{i(2)}\psi$ or $v^i\partial_t(\epsilon N_2^{i(2)}\psi)$. A glance at the form of $N_2^{i(2)}$ (B.4) shows us that the new terms in $(N_2^{i(3)}, M_2^{(3)})$ induced by this variation never have the right gamma matrix structure to source the spin-spin coupling.

The same is true for the variation of schematic form

$$g_2^{(2)}\psi^3 M_0^{(2)}\epsilon$$

because $M_0^{(2)} \sim \gamma^k v^k$, and $g_4^{(2,1)} = 0$. It is less obvious, but also true, that the variation

$$2g_2^{(2)} v^2 x^i v^j \psi\gamma^{ij} M_2^{(2)}\epsilon$$

gives rise to no relevant sources for $(N_2^{i(3)}, M_2^{(3)})$ although it does source the $g_4^{(3,1)}$ coupling. This leaves the variation

$$\delta(g_2^{(2)} v^2 x^i v^j)\psi\gamma^{ij}\psi$$

using $N_0^{i(2)}$. Almost all the terms in this variation play no role except for

$$\cdots + g_2^{(2)} v^2 x^i \left\{ i g_0^{(2)} v^2 (\epsilon \gamma^j \dot{\psi}) \right\} (\psi \gamma^{ij} \psi).$$

This term sources $M^{(3)}$ giving a contribution,

$$M_2^{(3)} \epsilon = \cdots - g_2^{(2)} g_0^{(2)} v^4 x^i (\gamma^j \epsilon)(\psi \gamma^{ij} \psi)\,. \tag{2.27}$$

We now have all the ingredients needed to study the generation of the 3-loop spin-spin coupling at $O(v^8)$.

2.5 A 3-loop prediction at $O(v^8)$

Let us finally put together all the sources for $g_4^{(4,1)}$. The $O(v^4)$ terms are generated at one-loop. By explicit computation [27], we know that

$$g_0^{(2)} = -\frac{15}{16} \frac{1}{r^7}\,. \tag{2.28}$$

This generates $g_2^{(2)}$ via (2.14). The other input we require is the value of $g_3^{(0)}$. This involves knowledge of the homogeneous solution at $O(v^6)$. In this particular case, we know there is no independent homogeneous solution. All terms at $O(v^6)$ are determined by terms at $O(v^4)$. Again from an explicit two-loop computation, we know that [25]

$$g_0^{(3)} = -\frac{225}{64} \frac{1}{r^{14}}\,. \tag{2.29}$$

This again determines $g_2^{(3)}$ via (2.25). Lastly, $g_4^{(3,1)}$ is determined by (2.24). All the unknown functions are now fixed.

The first source contribution comes from $\delta^3 S_2$. The variation,

$$2 g_2^{(2)} v^2 x^i v^j \psi \gamma^{ij} M_2^{(3)} \epsilon,$$

acts as a source for $g_4^{(4,1)}$ when we consider the terms

$$M_2^{(3)} = \cdots + \{ g_2^{(3)} - 2 i x^2 \tilde{g}_4^{(3,1)} - g_2^{(2)} g_0^{(2)} \} v^4 (x^i \psi \gamma^{ij} \psi)(\gamma^j \epsilon).$$

There are no other sources from S_2. From $\delta^2 S_3$, we obtain a similar contribution

$$2 g_2^{(3)} v^4 x^i v^j \psi \gamma^{ij} M_2^{(2)} \epsilon,$$

where the only relevant term involves

$$M_2^{(2)} = \ldots + g_2^{(2)} v^2 (x^i \psi \gamma^{ij} \psi)(\gamma^j \epsilon).$$

There is one other contribution from $\delta^2 S_3$ coming from

$$4 g_4^{(3,1)} v^4 x_i x_j (\psi \gamma^{ik} \psi)(\psi \gamma^{jk} M_0^{(2)} \epsilon),$$

where

$$M_0^{(2)}\epsilon = 3g_0^{(2)} v^2 v^k \gamma^k \epsilon\,.$$

Putting together all these contributions gives us the prediction,

$$4g^{(4,1)} = 2g_2^{(2)} \left\{ g_2^{(3)} - 2ix^2 \tilde{g}_4^{(3,1)} - g_2^{(2)} g_0^{(2)} \right\} - 2g_3^{(2)} g_2^{(2)} - 12 g^{(3,1)} g_0^{(2)}\,. \tag{2.30}$$

More important than the specific numerical value is the claim that this coupling is determined in terms of more relevant interactions, and that the coupling is 3-loop exact both perturbatively and non-perturbatively.

2.6 Generalizing the argument

This argument generalizes in two ways. First, it should be clear that by repeating the argument, one learns about the sources for the $(n-1)$-loop the spin-spin coupling at $O(v^{2n})$. While we can predict that this coupling is induced at this loop order by more relevant interactions, we need more information to determine the exact value of the coupling. At the moment, it is not sufficient to know just the $O(v^4)$ terms. In particular, we need to know about $g_2^{(m)}$ for all $m < n$. Or equivalently, we need to know about $g_0^{(m)}$ for all $m < n$. The two couplings are related by (2.14) up to source terms. It might be the case that supersymmetry determines all $(n-1)$-loop terms at $O(v^{2n})$ (in a way outlined in the introduction), but that question is beyond the scope of this analysis. What seems clear is that there will be more couplings beyond the spin-spin coupling (1.3) determined by more relevant sources.

The second way the argument generalizes explains the existence of rays in figure 1 with slopes other than 1. Most of our previous discussion is not special to the $O(v^4)$ terms in any way. Suppose at $O(v^{2n})$, there is a new m-loop contribution to v^{2n}. In terms of figure 1, this corresponds to an anchor point at coordinates (v^{2n}, m). By definition, nothing can source these terms so we can conclude that

$$g_4^{(n,1)} = 0 \tag{2.31}$$

at m-loops. However, a spin-spin coupling is generated at $(m+1)$-loops at $O(v^{2n+2})$. This comes about via the variation of the spin coupling,

$$2g_2^{(n)} v^{2n-2} x^i v^j \psi \gamma^{ij} M_2^{(2)} \epsilon\,,$$

using the $O(v^4)$ correction to the supersymmetry transformations. So the coefficient of this induced spin-spin coupling is completely determined. This is the reason a ray of slope 1 emanates from each anchor point. Indeed the same is true if we consider the variation,

$$2g_2^{(n)} v^{2n-2} x^i v^j \psi \gamma^{ij} M_2^{(k)} \epsilon\,,$$

for any k. This is the reason that all possible rays (one for each anchor point) emanate from each anchor point, and the reason for the intricate web of induced couplings depicted in figure 1.

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A. Fermion structures

A.1 Two fermion structures

Using the simplifications described in section 2.2, we can determine the form of the action (2.9) completely. Each real fermion transforms in the **16** of $Spin(9)$. We note that

$$\mathbf{16} \wedge \mathbf{16} = [2] \oplus [3]\,,$$

where $[n]$ refers to the antisymmetric n-form representation. The basic fermion bilinears are therefore,

$$\psi\gamma^{ij}\psi, \quad \psi\gamma^{ijk}\psi\,.$$

We will call these 2-gamma and 3-gamma structures, respectively. All of the couplings in the lagrangian are constructed from these building blocks contracted with x and v. For example, the only possible 2 fermion structure is

$$x^i v^j \psi\gamma^{ij}\psi\,. \tag{A.1}$$

A.2 Four fermion structures

To construct higher fermion structures, we will make use of some simplifying identities. The basic Fierz identities found in [1, 4] teach us that

$$\begin{aligned}(\psi\gamma^{ij}\psi)(\psi\gamma^{ij}\psi) &= 0\,,\\ (\psi\gamma^{ijk}\psi)(\psi\gamma^{ijk}\psi) &= 0\,,\\ (\psi\gamma^{ij}\psi)(\psi\gamma^{ijk}\psi) &= 0\,,\end{aligned} \tag{A.2}$$

and that $(\psi\gamma^{ijk}\psi)(\psi\gamma^{imn}\psi)$ can be expressed entirely in terms of 2-gamma structures $(\psi\gamma^{pq}\psi)$. When combined with CPT invariance, which acts as complex conjugation while sending

$$x \to -x, \qquad t \to -t\,,$$

these constraints will allow us to express all 4 fermion structures in terms of 2-gamma structures alone.

To see this, note that the 4 fermion structure appears with a real function of (x, v) if we want a hermitean coupling. It must therefore be even in x, and by Spin(9) invariance, even in v. The only possible structure involving a 3-gamma bilinear has the form,

$$(\psi\gamma^{ijk}\psi)(\psi\gamma^{il}\psi)\,.$$

However, this will be odd in either x or v. So we can restrict to structures built from 2-gamma bilinears.

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There are three possible terms which all appear in the supersymmetric completion of v^4,

$$\begin{aligned}&x_i x_j (\psi\gamma^{ik}\psi)(\psi\gamma^{jk}\psi)\,,\\&v_i v_j (\psi\gamma^{ik}\psi)(\psi\gamma^{jk}\psi)\,,\\&x_i x_j v_k v_l (\psi\gamma^{ik}\psi)(\psi\gamma^{jl}\psi)\,.\end{aligned}\tag{A.3}$$

This is no surprise since the most general fermion structure requires enough velocities so that we can, if we wish, attach a single velocity to any fermion bilinear. With 2 bilinears, this constraint means we need 2 velocity factors which is precisely the number available at order $n=2$.

A.3 Beyond four fermions

I cannot resist pushing this discussion a little further. How many 6 fermion couplings exist? If we want a hermitean term in the action then each 6 fermion coupling appears with an imaginary function of x and v. Invariance under CPT then implies that the coupling is odd in x. Lastly, these couplings appear with a v^{2n-3} factor so the coupling must also be odd in v. These are exactly the characteristics enjoyed by the 2 fermion coupling $(A.1)$.

It is not hard to check that any coupling constructed from a 3-gamma structure either fails to satisfy these constraints, or can be rewritten in terms of 2-gamma structures. The possible 2-gamma structures must be odd in both x and v so they can have 1 or 3 factors of x or v. This gives the following 4 possibilities (two of the structures are simply $x \leftrightarrow v$ exchanges)

$$\begin{aligned}&x_i v_j (\psi\gamma^{ik}\psi)(\psi\gamma^{jl}\psi)(\psi\gamma^{kl}\psi)\,,\\&x_i x_j x_k v_p (\psi\gamma^{ip}\psi)(\psi\gamma^{jl}\psi)(\psi\gamma^{kl}\psi)\,,\\&v_i v_j v_k x_p (\psi\gamma^{ip}\psi)(\psi\gamma^{jl}\psi)(\psi\gamma^{kl}\psi)\,,\\&v_i v_j v_k x_p x_q x_r (\psi\gamma^{ip}\psi)(\psi\gamma^{jq}\psi)(\psi\gamma^{kr}\psi)\,.\end{aligned}\tag{A.4}$$

How about 8 fermion couplings? These couplings must be even in both x and v. Let us first consider the 3-gamma structures. The only way a 3-gamma structure can appear is in the combination

$$(\psi\gamma^{ijk}\psi)(\psi\gamma^{il}\psi)\,.$$

Let us denote this structure by $(3+2)$. It has 3 free indices. The first possible 3-gamma structure is $(3+2)+(2+2)$, but this combination has an odd number of free indices since any contractions remove indices in pairs. It therefore cannot be even in both x and v, and is ruled out. The other possibility is $(3+2)+(3+2)$. In this case, we are not allowed to contract the 3-gamma structures together since the result can be expressed in terms of 2-gamma structures, but there are other possible contractions. It turns out, however, that all the other contractions vanish.

The remaining possibility is no contractions. In this case, we need to consider the square of

$$x^j v^k v^l (\psi\gamma^{ijk}\psi)(\psi\gamma^{il}\psi)\,,$$

or the same structure with $x \leftrightarrow v$. This coupling does not seem to vanish. It also does not seem reducible in an obvious way to a product of 2-gamma structures.

The remaining possibilities are built from 2-gamma bilinears. A list of structures contains

$$
\begin{aligned}
&(\psi\gamma^{ij}\psi)(\psi\gamma^{jk}\psi)(\psi\gamma^{kl}\psi)(\psi\gamma^{li}\psi)\,,\\
&x^i x^m(\psi\gamma^{ij}\psi)(\psi\gamma^{jk}\psi)(\psi\gamma^{kl}\psi)(\psi\gamma^{lm}\psi)\,,\\
&v^i v^m(\psi\gamma^{ij}\psi)(\psi\gamma^{jk}\psi)(\psi\gamma^{kl}\psi)(\psi\gamma^{lm}\psi)\,,\\
&x^i x^k x^l x^m(\psi\gamma^{ij}\psi)(\psi\gamma^{jk}\psi)(\psi\gamma^{lp}\psi)(\psi\gamma^{pm}\psi)\,,\\
&v^i v^k v^l v^m(\psi\gamma^{ij}\psi)(\psi\gamma^{jk}\psi)(\psi\gamma^{lp}\psi)(\psi\gamma^{pm}\psi)\,,\\
&x^i x^k v^l v^m(\psi\gamma^{ij}\psi)(\psi\gamma^{jk}\psi)(\psi\gamma^{lp}\psi)(\psi\gamma^{pm}\psi)\,,\\
&\{x^j v^k v^l(\psi\gamma^{ijk}\psi)(\psi\gamma^{il}\psi)\}^2\,, \qquad\qquad (A.5)\\
&\{v^j x^k x^l(\psi\gamma^{ijk}\psi)(\psi\gamma^{il}\psi)\}^2\,. \qquad\qquad (A.6)
\end{aligned}
$$

Note that many of these structures are simply related by $x \leftrightarrow v$. If the last 2 structures of $(A.5)$ are truly independent (as they appear to be) then they are likely to give rise to new non-generic couplings in the action.

It would be unpleasant if we now had to consider 10 fermion couplings and onwards. Fortunately, m fermion couplings are related by Hodge duality to $16-m$ fermion couplings, so we have already classified all possible terms! To set conventions, let us define the Hodge dual of an m fermion term, $T_{a_1\cdots a_m}\psi_{a_1}\ldots\psi_{a_m}$ by

$$
*T_{a_{m+1}\cdots a_{16}} = \epsilon_{a_1\cdots a_m a_{m+1} a_{16}} T^{a_1\cdots a_m}\,. \tag{A.7}
$$

B. Corrections to the SUSY transformations

In this appendix, we will list the homogeneous form for N_2^i and M_2 obtained by studying the variation of the action (2.9) into 3 fermions. This generalizes the results appearing in [22]. By studying the terms that involve no a or $\dot\psi$, we see that

$$
ig_2^{(n)} + 4g_4^{(n,2)} - 2\tilde g_4^{(n,1)} = 0\,, \tag{B.1}
$$

$$
4g_4^{(n,1)} + 4g_4^{(n,3)} + \frac{i}{r}\{g_2^{(n)}\}' = 0\,, \tag{B.2}
$$

$$
g_4^{(n,1)} = 0\,. \tag{B.3}
$$

By studying the $(a,\dot\psi)$ terms, we determine the homogeneous solution for N_2^i and M_2

$$
\begin{aligned}
\epsilon N_2^i\psi =\; & (4-4n)\tilde g_4^{(n,1)} v^{2n-4} x^2 v^i (x^l\psi\gamma^{lk}\psi)(\epsilon\gamma^k\psi) + \\
& + (8-4n)\tilde g_4^{(n,3)} v^{2n-6} x^2 v^i (x^l v^k \psi\gamma^{lk}\psi)(v^p\epsilon\gamma^p\psi) - \\
& - 2\tilde g_4^{(n,3)} x^2 v^{2n-4}(x^l\psi\gamma^{li}\psi)(v^p\epsilon\gamma^p\psi) - 2\tilde g_4^{(n,3)} x^2 v^{2n-4}(x^l v^k\psi\gamma^{lk}\psi)(\epsilon\gamma^i\psi)\,, \qquad (B.4)\\
M_2\epsilon =\; & -2i\tilde g_4^{(n,1)} x^2 v^{2n-2} x^i\left[2(\gamma^{ik}\psi)(\psi\gamma^k\epsilon) + (\psi\gamma^{ik}\psi)(\gamma^k\epsilon)\right] - \\
& - 2i\tilde g_4^{(n,3)} x^2 v^{2n-4} x^i v^k v^l\left[2(\gamma^{ik}\psi)(\psi\gamma^l\epsilon) + (\psi\gamma^{ik}\psi)(\gamma^l\epsilon)\right] + \\
& + (2n-2)g_2^{(n)} v^{2n-4}(x^i v^j\psi\gamma^{ij}\psi)(v^l\gamma^l\epsilon) + g_2^{(n)} v^{2n-2}(x^i\psi\gamma^{ij}\psi)(\gamma^j\epsilon)\,. \qquad (B.5)
\end{aligned}
$$

This homogeneous solution will, in general, be modified by sources generated by more relevant couplings in the effective action.

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